Real continuum

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Abstract

Some physical consequences of the negation of the continuum hypothesis are considered. It is shown that quantum and classical mechanics are component parts of the multicomponent description of the set of variable infinite cardinality. Existence and properties of the set follow directly from the independence of the continuum hypothesis. Particular emphasis is laid on set-theoretic aspect.

1 Introduction. One more aspect of the continuum problem

Space of classical physics is the continuous set. All the physical objects, including fields, can be removed from spatial continuum at least theoretically. In quantum physics, the complete elimination of imbedded structure from continuum is not possible: it is the inseparable part of the quantum vacuum. Real spacetime continuum contains some special microscopic ingredient. This very complicated "insertion unit" is, in fact, result of size reduction of the primitive classical continuous set over some degree of smallness. But although real space has several absolute orders of magnitude (scales) at which different dynamical rules are dominating, the basic structure of mathematical continuum (the set of all real numbers) does not have any criterion of size.

It is well known that any interval of continuum has the same number of points as the entire set of the real numbers. Moreover, arbitrarily small, even infinitesimal, interval of the real line has the same number of points as all the continuous universe of any number of dimensions.

In order to make properties of formal continuum conform to variable properties of real continuum and avoid the insertion structure, we shall correct this implausible superhomogeneity of the set of the real number.

Equivalence of all continuous intervals (without loss of generality we shall consider one-dimensional case) should be replaced by the following realistic dependence pattern of cardinality of an interval on its size: When the interval is large enough, its cardinality is close to cardinality of continuum (this is just mere establishment of the fact), i.e., all the intervals of space, down to some degree of smallness, are practically equipotent. Decrease of the number of points of the intervals is imperceptible.

This means that all such intervals have regular length, since length implies one-to-one correspondence between the points of any of the intervals l_{large} and the set of the real numbers R and, thus, may be regarded as manifestation of the equivalence

$$l_{large} \leftrightarrow R. \tag{1}$$

When difference between cardinality of an interval and cardinality of continuum become substantial, it should adversely affect length of the interval. Strictly speaking, length, as the establishment of equivalence $l_{large} \leftrightarrow R$, should vanish. Formally, a small interval such that $|l_{small}| < |R|$ turns to point. But it is natural to expect existence of a transition region: length of the noticeably non-continuous interval must show some irregularity, Measurements (direct or indirect) cannot give a unique stable real number.

Since any infinite set should be equivalent to its proper subset, infinite number of points decreases by steps: only some infinite "portion" of points changes the total number (makes the set non-equivalent to the initial one). By this reason, we can get only infinite number of points as a final result (finite set is not equivalent to any of its proper subset). In other words, real continuum is not infinitely divisible, i.e., there exists the infinite minimal set (greatest lower bound) instead of the minimal length.

If the infinite number of points of a continuous interval can decrease and the set of the points of sufficiently small interval becomes non-equivalent to the set of the real numbers, then the continuum hypothesis is false, i.e., the above assumption may be regarded as a form of the negation of the continuum hypothesis (CH) which seemingly contradicts to its independence. Note that the independence of CH can be established only in the framework of certain formal system, whereas the continuum problem had been stated before creation of any axiomatic set theory. Informally, the independence is not a final solution. Such a solution should either determine status of the set of intermediate cardinality or show that the continuum problem is meaningless.

Fortunately, there is a unique status of the intermediate set consistent with the independence of CH: since, by definition, continuum R should contain the subset of intermediate cardinality M such that $|N| < |M| < |R|$, where N is a set of the natural numbers, the independence of CH means that for any real number $x \in R$ the statement $x \in M \subset R$ is undecidable, i.e., the intermediate subset cannot be extracted from continuum. In other words, since non-existence of the set clearly contradicts the independence of CH, the only possible understanding is inseparability of the subset. Reason for this confinement should be investigated.

However, postponing investigation of the reason, we can use the independence of CH in order to get quite rigorous result.

2 From the continuum problem to path integrals

2.1 Thesis

The latent status is the only definite status of the set of intermediate cardinality that is consistent with the generally accepted solution of the continuum problem. However, it is necessary to know that standard Zermelo-Fraenkel set theory (ZF) gives correct description of the notion of set, i.e., we need set-theoretic analog of Church's thesis in order to be sure that we have reliable solution of the continuum problem independent of the concrete formalization of the concept of set.

2.2 Maps

Consider the maps of the intermediate set I to the sets of real numbers R and natural numbers N:

$$N \leftarrow I \rightarrow R. \tag{2}$$

Let the map $I \rightarrow N$ decompose I into the countable set of mutually disjoint infinite subsets: $\cup I_n = I$ $(n \in N)$. Let I_n be called a unit set. All members of I_n have the same countable coordinate n.

Consider the map $I \rightarrow R$. By definition, continuum R contains a subset M equivalent to I, i.e., there exists a bijection

$$f : I \rightarrow M \subset R. \tag{3}$$

This bijection reduces to separation of the intermediate subset M from continuum. For example, the separation of three real numbers is equivalent to the bijection $(1, 2, 3) \rightarrow R$. If we do not use any rule for the separation, we get the random (arbitrarily chosen) numbers (r_1, r_2, r_3). This randomness is not of principle because there are many rules for separation of three numbers as well as of any finite or countably infinite number of the real numbers. But in the case of the intermediate set, we, in principle, do not have a rule for separation of any subset with this number of members, since any separation rule for such a subset that can be expressed in ZF is a proof of existence of the intermediate set and, therefore, contradicts the independence of the continuum hypothesis.

Thus we, in principle, do not have a rule for assigning a definite real number to an arbitrary point s of the intermediate set. Hence, any bijection can take the point only to a random real number.

2.3 Intermediate set

This does not mean that the intermediate set consist of random numbers or that the members of the set are in any other sense random. Each member of the set of intermediate cardinality equally corresponds to all real numbers until the mapping has performed operationally. After the mapping, the concrete point gets the random real number as its coordinate in continuum, i.e., we get the probability $P(r)dr$ of finding the point $s \in I$ about r.

The independence of the continuum hypothesis is proved by construction of models of ZF with and without the intermediate set. In contrast to the model without the set which is the "smallest set theory" consisting only of constructible sets, the model with the "set of intermediate cardinality" is some unnatural extension of set theory or rather its distortion. Of course, the real intermediate set is not constructed. These models establish that the existence of the set of intermediate cardinality does not affect the formalized properties and interrelations of sets. Thus one cannot state that the intermediate set does not exist but the effect of its presence is absent. The set is "ZF-imperceptible".

2.4 Coordinates

Each member of the set of intermediate cardinality equally corresponds to all real numbers until the mapping has performed operationally. After the mapping, a concrete point gets random real number as its coordinate in continuum. Thus the point of the intermediate set has two coordinates: a definite natural number and a random real number:

$$s : (n, r_{random}). \tag{4}$$

Only the natural number coordinate gives reliable information about relative positions of the points of the set and, consequently, about size of an interval. But the points of a unit set are indistinguishable.

2.5 Transition

Consider probability $P(b,a)$ of finding the point s at b after it was found at a. The interval (a,b) defines parameterization of the coordinates of the point s. Let the parameter be denoted by t:

$$s(t) : [n(t), r(t)], \tag{5}$$

where $a < t < b$, $a = t_a = r(t_a)$, $b = t_b = r(t_b)$. Note that $n = n(r)$ does not exist: since $r = r(n)$ is random number, the inverse function is meaningless.

We have the right to consider the behavior of the point between a and b and to identify this parameter with time.

Since the point at any t corresponds to all real numbers simultaneously, it corresponds to all continuous random sequences of the real numbers (paths) $r(t)$. The elemental events are not mutually exclusive and, therefore,

$$P(b,a) \neq \sum_{all\, r(t)} P[r(t)], \tag{6}$$

where $P[r(t)]$ is the probability of finding the point s at any t on some arbitrary path (a continuous sequence of random real numbers) $r(t)$.

2.6 Probability

We cannot compute the probability $P(b, a)$ in the ordinary way, i.e., by summing or integration of $P[r(t)]$. In order to overcome this obstacle, it is most natural to introduce some additive functional $\phi[r(t)]$ such that

$$P[r(t)] = \mathcal{P}\{\phi[r(t)]\} \tag{7}$$

and

$$P[b, a] = \mathcal{P}(\sum_{all\, r(t)} \phi[r(t)]). \tag{8}$$

In other words, we shall compute the non-additive probability from the additive functional by a simple rule. It is clear that the dependence should be non-linear:

$$\mathcal{P}(\sum_{all\, r(t)} \phi[r(t)]) \neq \sum_{all\, r(t)} \mathcal{P}\{\phi[r(t)]\}. \tag{9}$$

We may choose the dependence arbitrarily. The simplest non-linear dependence is the square dependence:

$$\mathcal{P}[r(t)] = |\phi[r(t)]|^2. \tag{10}$$

The function ϕ, obviously, depends on $n(t)$. Since at any t the point equally corresponds to all the real numbers, it equally corresponds to all $r(t)$. This symmetry is of principle because it follows directly from the independence of the continuum hypothesis. As a result of the symmetry, all $r(t)$ are equiprobable: $P[r(t)]$ does not depend on $r(t)$, therefore, modulus of ϕ is constant and $n(t)$ may appear only in its phase. At the same time, it is necessary to ensure invariance of the $P[r(t)]$ under shift in N:

$$|\phi[r(t), n(t)]|^2 = |\phi[r(t), n(t) + const]|^2. \tag{11}$$

Hence, the function ϕ is of the following form:

$$\phi[r(t)] = const\, e^{2\pi i\, F[n(t)]}, \tag{12}$$

where $F[n(t)]$ is some real-valued additive functional of $n(t)$.

It is clear that $F[n(t)]$ is directly proportional to the other additive functional of $n(t)$: the length m of the countable path $n(t)$. Let us put:

$$F[n(t)] = m. \tag{13}$$

Then we get

$$\phi[r(t)] = const\, e^{2\pi i m} \tag{14}$$

and

$$P(b, a) = |\sum_{all\, r(t)} \mathrm{const}\, e^{2\pi i m}|^2, \tag{15}$$

i.e., the probability $P(a,b)$ of finding the point s at b after finding it at a satisfies the conditions of Feynman's approach (section 2-2 of [2]) for $S/\hbar = 2\pi m$. Therefore,

$$P(b,a) = |K(b,a)|^2, \tag{16}$$

where $K(a,b)$ is the path integral (2-25) of [2]:

$$K(b,a) = \int_a^b e^{2\pi i m} \mathcal{D}r(t). \tag{17}$$

Since Feynman does not essentially use in Chap.2 that $S/\hbar$ is just action, the identification of $2\pi m$ and $S/\hbar$ may be postponed.

2.7 Principle of least action, quantum of action, and mass

In section 2-3 of [2] Feynman explains how the principle of least action follows from the dependence

$$P(b,a) = |\sum_{all\, r(t)} \mathrm{const}\, e^{(i/\hbar)S[r(t)]}|^2. \tag{18}$$

This explanation may be called "Feynman's correspondence principle". We can apply the same reasoning to Eq.(15) and, for very large m, get "the principle of least m". This also means that for large m the point s has a definite stationary path and, consequently, a definite continuous coordinate. In other words, the interval of the intermediate set with the large countable length m is sufficiently close to continuum (let the interval be called macroscopic), i.e., cardinality of the intermediate set depends on its size.

For sufficiently large m,

$$F[n(t)] = \int_a^b dm(t) = \int_a^b \frac{dm(t)}{dt}\, dt. \tag{19}$$

Note that $m(t)$ is a step function and its time derivative is almost everywhere exact zero. But for sufficiently large increment $dm(t)$ the time derivative $\frac{dm}{dt} = \dot{m}(t)$ makes sense as non-zero value.

The function $m(t)$ may be regarded as some function of $r(t)$: $m(t) = \eta[r(t)]$. It is important that $r(t)$ is not random in the case of large m. Therefore,

$$\int_a^b dm(t) = \int_a^b \frac{d\eta}{dr}\, \dot{r}\, dt = \min, \tag{20}$$

where $\frac{d\eta}{dr}\, \dot{r}$ is some function of r, $\dot{r}$, and t. This is a formulation of the principle of least action (note absence of higher time derivatives than $\dot{r}$), i.e., large m can be identified with action.

Recall that this identification is valid only for very large $dm = \dot{m}dt$, i.e., for sufficiently fast points. In fact, this is a qualitative leap: action is not the length of the countable path but some new function. We get a new characteristics of the point and a new law of its motion.

Since the value of action depends on units of measurement, we need a parameter h depending on units only such that

$$hm = \int_a^b L(r, \dot{r}, t)\, dt = S. \tag{21}$$

Finally, we may substitute $S/\hbar$ for $2\pi m$ in Eq.(17) and regard our consideration as a natural extension of Feynman's formulation of quantum mechanics.

The original Feynman's approach becomes more consistent with this extension because there is no need in physically meaningless segments of straight line or sections of the classical orbit between the points of the partition Eq.(2-19), Fig. 2-3 ([2]) by which Feynman constructs the sum over paths. There is also no need in existence of action from the very beginning.

Note that if time rate of change of cardinality (i.e., of the countable coordinate) is not sufficiently high, action vanishes: $\dot{m}(t)$ and, consequently, $dm = \dot{m}(t)dt$ is exact zero. This may be understood as vanishing of the mass of the point. Formally, mass is a consequence of the principle of least action: it appears in the Lagrangian of a free particle as its peculiar property [3]. Thus mass is somewhat analogous to air drag which is substantial only for sufficiently fast bodies.

3 Extra descriptions and extra dimensions

3.1 Intervals

If we reduce some interval of real continuum to the order of smallness, at which decrease in its cardinality is appreciable, we automatically reveal discrete properties of the interval: due to equivalence of any infinite set to its proper subset, cardinality decreases in steps, i.e., the interval becomes less similar to continuum (instability of its length) and more similar to the countable set (it gets one more length expressed by natural number).

Thus we get three kinds of the intervals:

large continuous intervals that have length as manifestation of their equivalence to the set of the real numbers;

insufficiently large submicroscopic intervals whose lengths are therefore unstable (the interdependence between instability of the interval length and its natural number length is the content of quantum mechanics);

small non-continuous microscopic intervals without length which are, actually, composite points.

3.2 Dimensionality

In order to keep inside certain cardinality, a shift should also has this cardinality. Therefore, one-dimensional intermediate axis splits into, at least, three non-equivalent "subaxes," i.e., immiscible substructures. The complete description is three-dimensional.

In this case, dimensionality is a classification of cardinalities. The classification with respect to length is the roughest (macroscopic) estimate of cardinality (yes, no, unstable). More precisely, length is an indication of degree of saturation of cardinality: saturated (continuum), unsaturated, close to saturation, respectively.

Saturation of cardinality is important because of the following paradox which is important for understanding of mechanical motion: when a point moves with high countable speed that may be regarded as a continuous variable, saturated cardinality of the path and its time rate of change are really constant. Cardinality does not change as in the case of "countably motionless" (massless) point. Very fast "vertical" motion turns to "horizontal".

Classical mechanics gives only one spatial dimension: the continuous coordinate. The independent natural number coordinate is replaced by the functional of the continuous coordinate and its time derivative (degeneration).

Quantum mechanics gives two coordinates: natural number and random real number.

The proper microscopic description does not give extra dimensions if the proper microscopic intervals are regarded as points. These composite points take part in classical and quantum-mechanical descriptions. However, since the microscopic intervals are essentially non-equivalent, they themselves are immiscible and form a quantity of different objects described by a hierarchy of theories. Therefore, description of the structure and transmutation of the intervals needs additional dimensions down to the single unit set. But these dimensions should manifest themselves rather as qualitative properties (charges) of the points (in other words, they are inherently "compact").

Thus we get three spatial dimensions (one macroscopic and two microscopic) and time in the one-dimensional case. It is interesting to note that in the three-dimensional case it gives ten spacetime dimensions just like in string theory. We also expect some unknown number of extra dimensions "inside the point".

Since the description of the one-dimensional intermediate set consist of "sections", which are on equal footing, the particular main laws, directions, and dimensions of the "sections" are equally valid. Thus we get parallel descriptions. These descriptions relate to different immiscible substructures of real continuum.

It is quite clear, regretfully post factum, that classical and quantum mechanics are mutually irreducible by any correspondence principle. Reducing Plank constant to zero, one cannot get classical mechanics. One can only make some operators commute.

Feynman's correspondence principle does not reduce classical mechanics to quantum mechanics but separates their fields of application. These fields are not only different scales of the same space (there are well-known macroscopic quantum phenomena). The distinction goes further because, due to non-equivalence of the subsets of different cardinalities, these subsets are closed under different equivalence relations (symmetry transformations) that leads to effect of separate dimensions and, consequently, directions. Therefore, these non-equivalent structures (ruled by different laws) become autonomous and immiscible.

4 Set theory and real continuum

4.1 Fission vs. construction

It is worthwhile to pay attention to the way of obtaining sets by fission of continuum. According to P. Cohen, continuum "can never be approached by any piecemeal process of construction" [1], therefore, it may be stated that members and subsets of continuum obtained by such a process are not true but at best imitation, i.e., the true members and subsets should be extracted from continuum itself by its fission.

4.2 Cardinality as a property

According to the separation axiom scheme, for any set and for any property expressed by some formula there exists a subset of the set, which contains only members of the set having the property. From the independence of CH it follows that we cannot express any property of the members of the intermediate set, i.e., any property we can formulate implies separation of either countable or continuous subset of continuum. This fact is the reason of the inseparability of the set of intermediate cardinality. The simplest way out is unexpected: to regard infinite cardinality itself as a property of the set members. Breaking real continuum, in which cardinality of any part depends on the size of this part, into intervals of lower cardinalities, we should consider each intermediate cardinality as an elementary property ("charge," power) of the corresponding interval.

Then we get one more aspect of the continuum problem: how many different properties (non-equivalent infinite fragments) can be derived from continuum?

From this standpoint, members of continuum are not real numbers, subsets of N, or zero length points but continuous intervals. The representation of the real numbers as nonterminating decimals implies infinite process of fission. On the contrary, intervals of continuum can be obtained by primitive finite procedures.

4.3 Intuition of continuum

We, obviously, have intuition of continuum but this intuition is not used in set theory perhaps because it does not coincide with the set of the real numbers: equivalence of an arbitrarily small interval to the entire set of the real numbers is clearly counter-intuitive.

At first sight, "axiom of continuum," stating existence of the structureless continuous whole, and formal scheme for its fission into equivalent and non-equivalent parts seem unavoidable in order to complete (balance) set theory: Zermelo-Fraenkel set theory has tools only for construction of sets. However, it is more important to have consistent factual picture independent of formalization. Formal results often need interpretations which sometimes constitute more difficult problems than formal solutions themselves. For instance, the continuum problem is much clearer when it is stated informally. It is interesting that formalists still are not sure that the problem and the problem of size of continuum generally make sense.

For the twentieth century, which was the century of search for the formal unity (universal formalization), a great number of statements in different areas of mathematics had been shown to be independent. This is quite explainable, taking into account that the present mathematics is inevitably macroscopic. Statements and concepts touching upon essentially microscopic aspects should be either independent, which shows absence of necessary information, or contradictory, which indicates attempt to unify distinct fields of reality. Sets are real objects and hardly can serve as unchangeable "mental units". Set theory with the intermediate set cannot be separated from, at least, microscopic (quantum) reality because set-theoretic and even logical notions appear to be connected with fine spatial structure.

Fundamental physics may be called "study of real continuum". The Gödel's incompleteness theorems, applied to the study, can be interpreted as impossibility of a unique unified theory of everything. The second incompleteness theorem points to the hierarchical structure of fundamental theories. A correct theory is not a limiting case of the next more exact theory. The correct theories form some structure which is directly related to the structure of continuum itself.

Since the complete description consist of interpenetrating parts governing by different rules, one can get formal contradiction as a real conflict of correct descriptions. In this case, elimination of contradictions in order to get consistent unified formal picture is inadmissible.

4.4 Cardinality and structure

Cantor's opinion that cardinality of a set is independent of nature and properties of its members is still regarded as indisputable. However, this is neither axiom nor theorem but only an observation on finite sets by default imposed on infinite ones. Note that, from some number of line segments, one can form the most complex structure (e.g. polygon) which may serve as a unique characteristic of the number, i.e., besides one-to-one correspondence, finite cardinality can be characterized by some structure.

Unlike a finite set, an infinite collection of members cannot be in disordered state or arbitrarily arranged. Since an infinite set is equivalent to its proper subset, it may be stated that any infinite set necessarily forms some symmetrical structure (asymmetrical arrangement is impossible).

Note that, the most symmetrical arrangement is the most probable one because such an arrangement has the greatest number of equivalent (symmetrical) states (the number of ways in which the arrangement can be produced: "thermodynamical probability" of the arrangement of the infinite set). Physically, this means that only the most symmetrical arrangement is stable, i.e., any infinite set forms the most symmetrical structure by itself. It is clear that symmetries (equivalence relations) of non-equivalent sets should be different. Hence, an infinite set can be characterized by type of symmetry and "charge" of its members.

We can formulate the following rule: any infinite set tends to form the most symmetrical structure determined by its cardinality.

In the world of finite sets, we need more bricks for a big palace than for a small house. The most complex building can play the role of the primitive symbol of the corresponding number of bricks. In the world of infinite sets, the role of such a symbol plays, figuratively, the smallest cabin, i.e., the simplest (most symmetrical) structure.

Whole continuum is regarded as elementary structureless object, to a certain degree, complementary to the empty set in present set theory which is obviously also without structure.

Absence of structure explains absolute homogeneity of the formal construction (the set of all real numbers) which is unconditionally identified with continuum.

In the case of macroscopic system, the rule of maximum symmetry works like the law of entropy increase: since any macroscopic system occupies continuous region, it tends to reproduce the absolutely homogeneous structureless continuous whole by disintegration of all macroscopic structures as inhomogeneities and making chaos. However, submicroscopic and proper microscopic objects successfully avoid this law because decrease of cardinality automatically entails structures, i.e., final states of these objects are structured (but non-stationary because symmetrization in multisructured system leads to dynamics). Hence, contrary to our expectation, we get increase of complexity of smaller objects. In other words, primitive spatial continuum really contains more complicated "insertion units;" the whole is simpler than its component parts. This conclusion is very strange indeed. Figuratively, complex microscopic structures are cut out of the whole "piece of wood" and then assembled into constructions ("Pinoccio making method").

Thus even global tendencies ("fates") of macroscopic, submicroscopic, and microscopic subworlds are different. For instance, atoms and particles never get old, while large molecules, e.g. protein, are already subjected to aging. Recall also that, in pure quantum systems, chaos, in classical sense, is absent. Quantum chaos can be defined only for semiclassical systems and this is rather theoretical possibility than phenomenon needing obligatory explanation.

Consequently, all the structures in our continuous universe have microscopic origin and are supported by the microscopic processes. Note that the only pure macroscopic object is classical vacuum which is really structureles. Geometry (and fields) requires presence of microscopic structures (matter). On the contrary, microscopic point-like objects are complex and can contain much more information than it is supposed.

If intermediate cardinalities are regular, then fission is irreversible: one cannot restore continuum or any intermediate interval as the union of the intervals of lower cardinalities. It is plausible because such a restoration is not a mere union of sets but synthesis of more homogeneous structure of higher cardinality from structures of lower cardinalities ("regeneration").

It is interesting that, practically, abstract set-theoretic regularity gives the notion of space itself in its visual sense (emptiness): structures, i.e., objects of smaller cardinalities (matter), cannot fill all the continuous superset.

4.5 Open-closed duality

Until the intermediate set is large enough to be regarded as continuous, the most symmetrical structures it can form are closed structures (loops): A small intermediate interval consist of a small finite number of unit sets. A finite set is not equivalent to any of its proper subsets, i.e., it has no natural symmetries (self-coincident moves) but the interval, as an infinite set, should take the most symmetrical form. The only way to get a natural symmetry is to form a loop. Note that the least number of unit sets for a loop is three.

It is also natural to expect existence of transition region, where closed interval is not stable enough (open-closed duality). Only sufficiently continuous interval can form stable open structure. However, such intervals are miscible with the continuum (the stable continuous intervals are members of continuum). Thus most of the strings should be closed; only more or less unstable open strings can be observed.

4.6 Real and virtual subsets

Since formal continuum (the set of the real numbers) differs from real one, the reason of the inseparability of the intermediate subset in the set of the real numbers is different from that in real continuum. Whereas the large intermediate set really contains such subset, the natural structure of formal continuum is only a carrier medium for the unstable manifestations of the members of the intermediate set. In the set of the real numbers, unstable members are artificial objects: they are not component parts but only possible formations in the structure of the set. However, one cannot state that exact continuum does not contain the intermediate subset, albeit its presence is rather virtual.

References

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[3] Landau L. D., Lifshitz E. M., Mechanics, Oxford; New York: Pergamon Press, 1976.